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Analytical solution for the electric potential due to a point source in an arbitrarily anisotropic half-space

PING LI and NORMAN F. UREN
Department of Exploration Geophysics, Curtin University of Technology, GPO Box U1987, Perth, WA 6845, Australia

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Abstract. A very large class of important theory and applications in geophysics requires analytical solutions for the determination of the electric potential due to a point source in an arbitrarily anisotropic half-space. In this paper, a very clear and simple solution to the half-space problem has been developed from consideration of an arbitrarily anisotropic whole space. For the first time, the method of images is used to generate the solution for an arbitrarily anisotropic three dimensional half-space. Based on traditional extreme value theory the image source point has been determined and calculated for the half-space case.

Key words: image method, anisotropic medium, electric potential, half-space, geophysics.

1. Introduction

The study of anisotropy in a half-space earth medium has captured the attention of many researchers. The reasons for this interest are not hard to perceive. Anisotropy is impossible to ignore, because it is so evident whenever field observations are analyzed. For instance, measurements of electrical conductivity, from rock samples in the laboratory or with borehole receivers, indicate that conductivity may vary by orders of magnitude with direction. Most properties which affect the rates of current density have been shown to be highly directionally variable. Although geophysicists have always been aware of anisotropy, they have had to use the classical prediction tools of geophysics, which have assumed some form of piecewise sectoral variation in formation parameters. There has been so far no analytical solution which can be used in a truly arbitrarily anisotropic medium.

So far, in anisotropic research in geophysics, a higher-order symmetry in the conductivity tensor has been considered by some workers, assuming that the principal axes of anisotropy are parallel to or perpendicular to the Earth's surface [1–6]. In 1986 [7] the electrical potential was given in terms of a conductivity tensor with purely diagonal nonzero terms. Although these are useful considerations in studying anisotropy effects in electrical measurements, they are too restrictive for many situations in the crust of the Earth. In this paper an arbitrarily anisotropic half-space potential will be developed analytically. This solution is closer to field situations than that obtained by assuming isotropy.

The concept of an image source was used in the pioneering studies [8–10], as well as in later works concerning the electric potential in layered homogeneous media. In his study, Asten [3] used the image-source method for the first time in an anisotropic medium where one of the principal axes of conductivity was parallel to the ground surface. Dellinger and Muir [11] gave a physical model for an anisotropic half-space image and Uren [12] showed how to locate the image point in the case of elliptically anisotropic seismic velocity. Elliptically anisotropic seismic velocity and electric conductivity may be described by second rank tensors and their

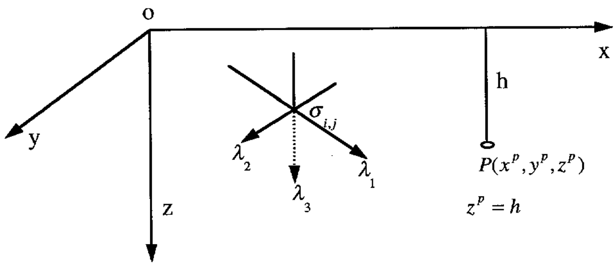

Figure 1. A point source embedded in an arbitrarily anisotropic half-space. The source point is at $P(x^p, y^p, z^p)$ and the λ_1, λ_2 and λ_3 are the principal axes of the conductivity tensor $\sigma_{i,j}$.

geometrical expression is an ellipse. Based on the symmetry of the ellipse the image-source point can be determined mathematically. In 1993, Lindell *et al.* [13], using an image-source method, computed the direct-current electrical potential in an azimuthally anisotropic half-space. In our paper we use a very simple mathematical method based on Uren's optical model to extend the image-source method to solve the problem of arbitrary principal axis orientation. Das [14] and Das and Li [15] have tried to use the method of images to solve this problem, but the solutions were impossible to use, because not only the fulfilment of the boundary conditions was not mentioned, but also the location of the image-source point was not given.

2. Mathematical description of the half-space system

Consider the potential due to a point source in an arbitrarily anisotropic half-space as depicted in Figure 1. We have chosen a Cartesian coordinate system (x, y, z) with the origin of coordinates on the surface of the Earth and the z-axis pointing vertically downward. We can formulate the problem mathematically by applying boundary conditions to an appropriate form of the solution to the governing equation. We assume Ohm's law relating the current density **J** to the electric field intensity **E** as

$$\mathbf{J}(x, y, z) = \sigma_{\mathbf{i},\mathbf{j}} \cdot \mathbf{E}(x, y, z), \tag{1}$$

where $\sigma_{\mathbf{i},\mathbf{j}}$ is the conductivity tensor which is symmetric and positive definite. For stationary electric fields the electric intensity is

$$\mathbf{E}(x, y, z) = -\nabla v(x, y, z), \tag{2}$$

where $v(x, y, z)$ is the electric potential.

Consider a point source for which we have

$$\nabla \cdot \mathbf{J}(x, y, z) = I\delta(x - x^p)\delta(y - y^p)\delta(z - z^p), \tag{3}$$

where δ is the Dirac delta function, I is the electrode current and (x^p, y^p, z^p) are the coordinates of the source point. Substituting (2) in (1) and then substituting (1) in (3) the following elliptic differential equation is obtained

$$\sigma_{1,1}\frac{\partial^2 v}{\partial x^2} + \sigma_{2,2}\frac{\partial^2 v}{\partial y^2} + \sigma_{3,3}\frac{\partial^2 v}{\partial z^2} + 2\sigma_{1,2}\frac{\partial^2 v}{\partial x \partial y} + 2\sigma_{1,3}\frac{\partial^2 v}{\partial x \partial z} + 2\sigma_{2,3}\frac{\partial^2 v}{\partial y \partial z}$$
$$= -I\delta(x - x^p)\delta(y - y^p)\delta(z - z^p). \tag{4}$$

In an anisotropic earth medium with a current point source, the governing partial differential equation is subject to the following boundary conditions.

(1) On the air-earth surface, the normal component (J_z) of current density **J** must be zero:

$$J_z = \sigma_{1,3}\frac{\partial v}{\partial x} + \sigma_{2,3}\frac{\partial v}{\partial y} + \sigma_{3,3}\frac{\partial v}{\partial z} = 0. \tag{5}$$

(2) At infinity the point-source potential must reduce to zero:

$$x, y \to \pm\infty, \qquad z \to \infty, \qquad v(x, y, z) \to 0. \tag{6}$$

3. Whole-space solution

The whole-space solution has been given in many papers in geophysical research, for example Eskola, [9, pp. 82–101]. However, these have led to a variety of different formulas. For convenience in this paper, a concise and explicit expression will be given.

The boundary condition of the governing (4) in the whole space is that for $x, y, z \to \pm\infty$, the potential $v(x, y, z) \to 0$. We apply the following rotation and translation coordinate transformations to (4) [16, pp. 70–76],

$$\begin{aligned}
\xi_1 &= \alpha_{1,1}(x - x^p) + \alpha_{1,2}(y - y^p) + \alpha_{1,3}(z - z^p),\\
\xi_2 &= \alpha_{2,1}(x - x^p) + \alpha_{2,2}(y - y^p) + \alpha_{2,3}(z - z^p),\\
\xi_3 &= \alpha_{3,1}(x - x^p) + \alpha_{3,2}(y - y^p) + \alpha_{3,3}(z - z^p).
\end{aligned} \tag{7}$$

Let the fourth, fifth and sixth terms in the left-hand side of (4) be zero and use v^p to express the potential for the whole space. Equation (4) then becomes

$$\lambda_1\frac{\partial^2 v^p}{\partial \xi_1^2} + \lambda_2\frac{\partial^2 v^p}{\partial \xi_2^2} + \lambda_3\frac{\partial^2 v^p}{\partial \xi_3^2} = -I\delta(\xi_1)\delta(\xi_2)\delta(\xi_2), \tag{8}$$

where λ_1, λ_2 and λ_3, are eigenvalues of the tensor $\sigma_{i,j}$, and $\alpha_{k,i}\lambda_k\alpha_{j,k} = \sigma_{i,j}$. We introduce new variables

$$\xi_1 = \eta_1\sqrt{\lambda_1}, \qquad \xi_2 = \eta_2\sqrt{\lambda_2}, \qquad \xi_3 = \eta_3\sqrt{\lambda_3}. \tag{9}$$

Equation (8) becomes Poisson's equation as

$$\frac{\partial^2 v^p}{\partial \eta_1^2} + \frac{\partial^2 v^p}{\partial \eta_2^2} + \frac{\partial^2 v^p}{\partial \eta_3^2} + = -\frac{I}{\sqrt{\lambda_1\lambda_2\lambda_3}}\delta(\eta_1)\delta(\eta_2)\delta(\eta_3), \tag{10}$$

where $\lambda_1\lambda_2\lambda_3$ is the third invariant of the conductivity tensor [17, pp. 121–132]. Using the property of matrix symmetry, we have

$$\frac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}} = (\det[\zeta_{i,j}])^{1/2}, \tag{11}$$

where $\zeta_{i,j}$ is the resistivity tensor which is the inverse of the conductivity tensor $\sigma_{i,j}$. Kevorkian, [18, pp. 58–59] gave the following solution to (10)

$$v^p = \frac{I'}{4\pi|\eta|}, \tag{12}$$

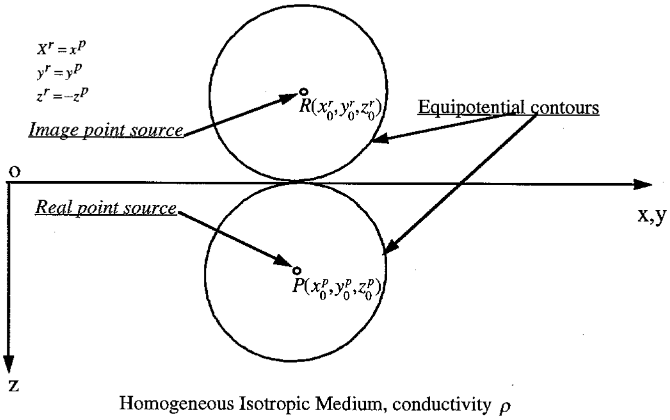

Figure 2. A point source $P(x^p, y^p, z^p)$ and its image-source point $R(x^r, y^r, z^r)$ in an isotropic half-space medium.

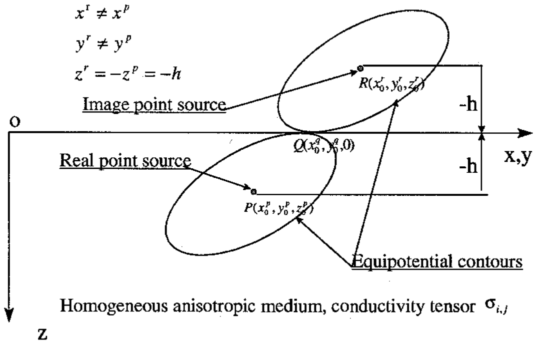

Figure 3. A point source $P(x^p, y^p, z^p)$ and its image-source point $R(x^r, y^r, z^r)$ in an anisotropic half-space medium.

where $I' = I(\det[\zeta_{i,j}])^{1/2}$ is constant. From (7) and (9), using $\alpha_{k,i}(\lambda_k)^{-1}\alpha_{j,k} = \zeta_{i,j}$, we have

$$|\eta| = [\zeta_{1,1}X^2 + \zeta_{2,2}Y^2 + \zeta_{3,3}Z^2 + 2\zeta_{1,2}XY + 2\zeta_{1,3}XZ + 2\zeta_{2,3}YZ]^{1/2}, \tag{13}$$

where $X = x - x^p$, $Y = y - y^p$ and $Z = z - z^p$. This is the final expression for the potential due to a point source in an arbitrarily anisotropic whole space.

4. The half-space solution

4.1. Image-source and general solution

In the whole homogeneous isotropic space the potential due to a point source is

$$v^p = \frac{I}{4\pi}\frac{1}{|\mathbf{r}|}, \tag{14}$$

where $|\mathbf{r}| = [X^2+Y^2+Z^2]^{1/2}$, and the equipotential surfaces are spherical. For the half-space, using an image point source R as indicated in Figure 2, the potential function is [16]

$$v(\mathbf{r}) = v^p(\mathbf{r}) + v^r(\mathbf{r}) = \frac{I}{4\pi}\left[\frac{1}{|\mathbf{r}|} + \frac{1}{|\mathbf{r}'|}\right], \tag{15}$$

where the first term is called the primary term and the second term is the reflection term. Also $|\mathbf{r}'| = [X_r^2 + Y_r^2 + Z_r^2]^{1/2}$, where $X_r = x - x^r$, $Y_r = y - y^r$, $Z_r = z - z^r$ and (x^r, y^r, z^r) are the coordinates of the image source point. Figure 2 clearly shows that $x^r = x^p$, $y^r = x^p$ and $z^r = -z^p$.

As for the isotropic half-space problem, we assume the solution for the potential in an arbitrarily anisotropic half-space to be

$$v = v^p(\eta) + v^r(\eta), \tag{16}$$

where $v^p(\eta)$ is the primary term that is the point-source potential in an arbitrarily anisotropic infinite medium as in (12), and $v^r(\eta)$ is the reflection term that is produced by the image source. Substituting (12) in (16), and noting the solution for the homogeneous medium given in (15), the following equation can be assumed to be the solution for a point source in an arbitrarily anisotropic medium half-space.

$$v = \frac{I'}{4\pi}\left[\frac{1}{|\eta|} + \frac{1}{|\eta'|}\right], \tag{17}$$

where $|\eta|$ is given in (13) and

$$|\eta'| = [\zeta_{1,1}X_r^2 + \zeta_{2,2}Y_r^2 + \zeta_{3,3}Z_r^2 + 2\zeta_{1,2}X_rY_r + 2\zeta_{1,3}X_rZ_r + 2\zeta_{2,3}Y_rZ_r]^{1/2}, \tag{18}$$

where $X_r = x - x^r$, $Y_r = y - y^r$, $Z_r = z - z^r$, and (x^r, y^r, z^r) are the coordinates of the image-source point as shown in Figure 3.

If (17) is the correct solution for an arbitrarily anisotropic half-space, it must satisfy the total reflection boundary condition, (5), and the sum of the potentials from source and image must vanish at infinity, (6). The image source location (x^r, y^r, z^r) also must be determined.

4.2. Image-point source

For convenience of demonstration, we first need to analyse some numerical results of the potential in an infinite space. Let an electric current point source be located at $P(x^p, y^p, h)$ in an arbitrarily anisotropic infinite medium with electrical conductivity tensors $\sigma_{i,j}^1$ and $\sigma_{i,j}^2$ (See Section 5). The equipotential contours on horizontal planes through the origin of coordinates for various values of h separations, *e.g.*, 5·0 m, 20·0 m, 40·0 m are as shown in parts (a), (b) and (c) of Figures 4 and 5. It is observed that with increasing vertical distance between the horizontal observation plane and the source, the confocal point, which is the location of the extreme value of potential, moves along the negative (in Figure 4) or the positive (in Figure 5) direction of the x and y axes.

To find the extreme point of the potential on the plane, $z = 0$, differentiate (12) with respect to the spatial coordinates x and y and then let the derivative be zero. The following linear equations can be obtained

$$\begin{aligned} \zeta_{1,1}(x - x^p) + \zeta_{1,2}(y - y^p) - \zeta_{1,3}h = 0, \\ \zeta_{2,1}(x - x^p) + \zeta_{2,2}(y - y^p) - \zeta_{2,3}h = 0. \end{aligned} \tag{19}$$

We solve these equations and use x^q and y^q to indicate the confocal point as

$$x^q = \frac{(\zeta_{2,2}\zeta_{1,3} - \zeta_{1,2}\zeta_{2,3})h}{\zeta_{2,2}\zeta_{1,1} - \zeta_{1,2}^2} + x^p, \qquad y^q = \frac{(\zeta_{1,1}\zeta_{2,3} - \zeta_{1,2}\zeta_{1,3})h}{\zeta_{2,2}\zeta_{1,1} - \zeta_{1,2}^2} + y^p. \tag{20}$$

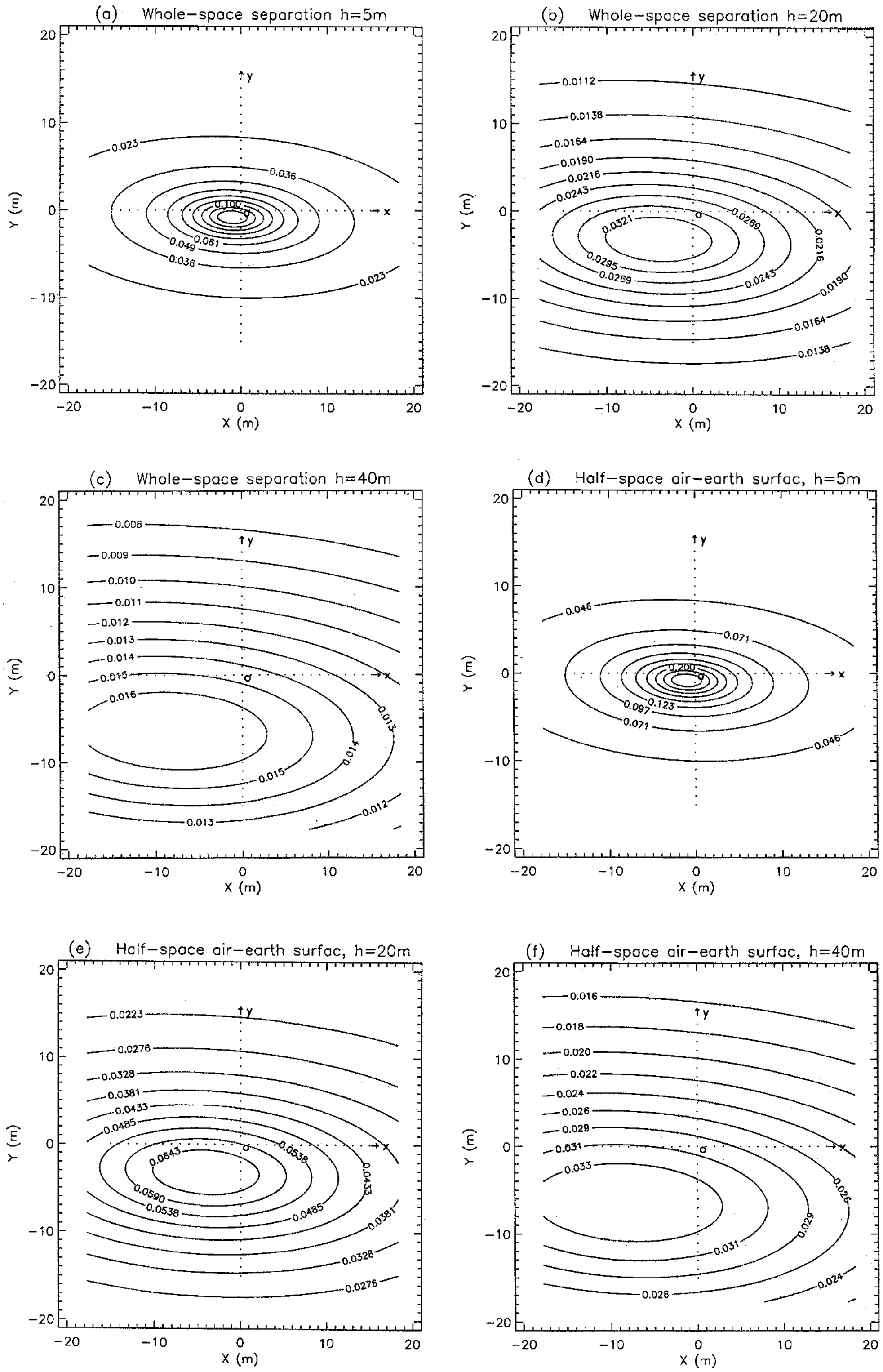

Figure 4. Tensor σ^1: Equipotential contours arising from a point source in an arbitrarily anisotropic medium. Figure 4(a, b and c) show the equipotentials in the whole space on a horizontal plane with the source depths at 5, 20 and 40 m. Figure 4(d, e and f) show the corresponding equipotential contours on a half-space air/earth surface.

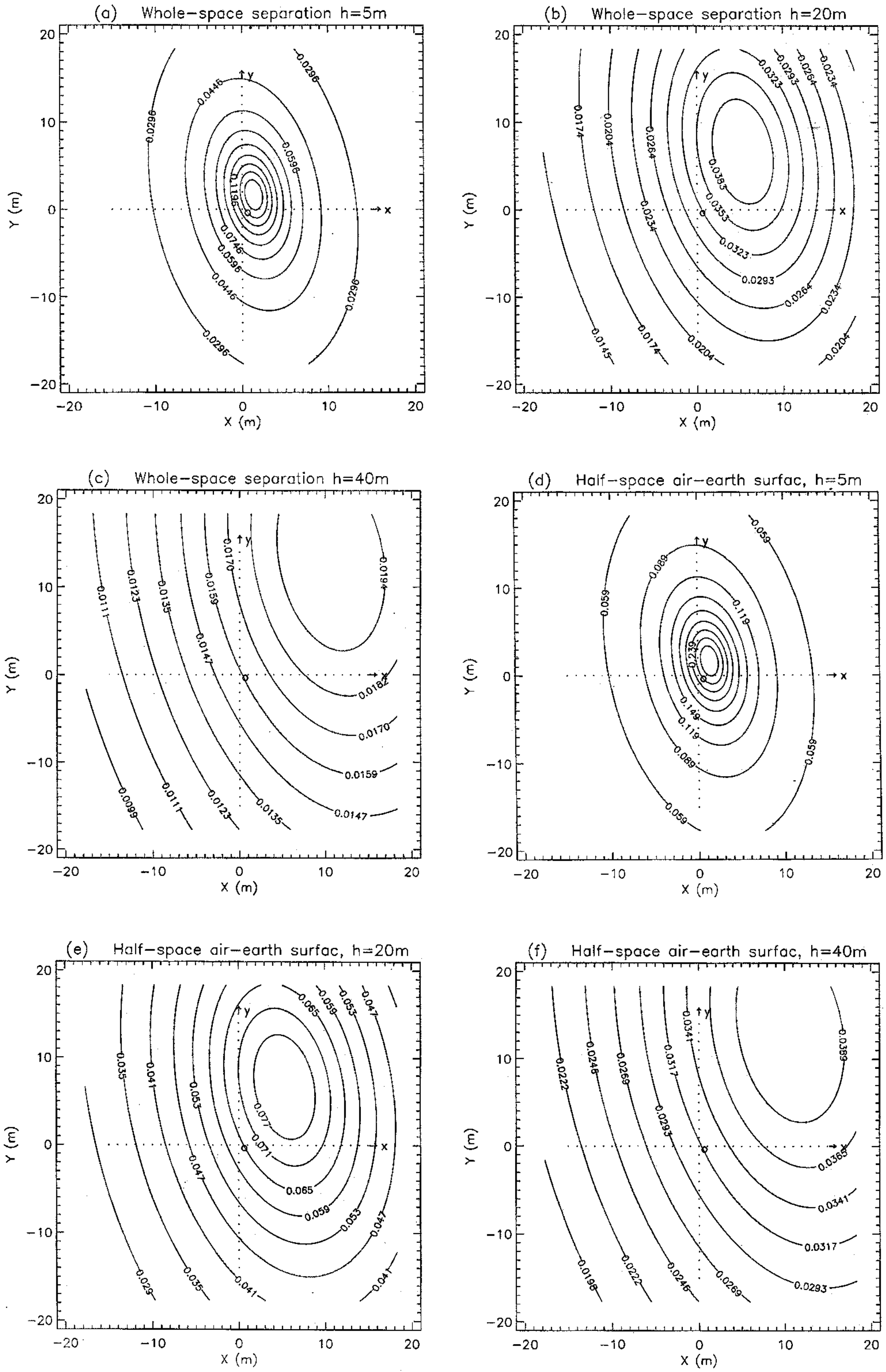

Figure 5. Tensor σ^2: Equipotential contours arising from a point source in an arbitrarily anisotropic medium. Figure 5(a, b and c) show the equipotentials in the whole space on a horizontal plane with the source depths at 5, 20 and 40 m. Figure 5(d, e and f) show the corresponding equipotential contours on a half-space air/earth surface.

If the medium is isotropic and homogeneous, and the source is at $P(x^p, y^p, h)$, then the image point will be $R(x^p, y^p, -h)$ and the extreme value of the potential on the surface $z = 0$ will be at $O(x^p, y^p, 0)$ as can be seen from Figure 2. Now because of the influence of anisotropy, the confocal point will be moved to point $Q(x^q, y^q, 0)$ as shown in Figure 3.

For convenience, the following relation, termed 'anisotropic divergence', can be defined,

$$\gamma_1 = \frac{x^p - x^q}{h} = \frac{\zeta_{2,2}\zeta_{1,3} - \zeta_{1,2}\zeta_{2,3}}{\zeta_{2,2}\zeta_{1,1} - \zeta_{1,2}^2}, \qquad \gamma_2 = \frac{y^p - y^q}{h} = \frac{\zeta_{1,1}\zeta_{2,3} - \zeta_{1,2}\zeta_{1,3}}{\zeta_{2,2}\zeta_{1,1} - \zeta_{1,2}^2}, \tag{21}$$

where γ_1 is the anisotropic divergence in the x-direction and γ_2 is the anisotropic divergence in the y-direction. We can see that they depend only on the resistivity tensor.

Considering the anisotropic divergence and the symmetry property of the ellipse, the image point source is assumed to be at point $R(x^r, y^r, z^r)$ (See Figure 3). The potential due to the image source is

$$v^r = \frac{I'}{4\pi}\frac{1}{|\eta'|}, \tag{22}$$

where the $|\eta'|$ as given in (18).

Because the resistivity tensor is directionally variable and spatially constant and the reflection plane is $z = 0$, it is obvious that the image source point must be located in the plane $z = -h$. This means that (Figure 3)

$$z^r = -h. \tag{23}$$

As before, differentiate (22) with respect to the spatial coordinates x and y and then let them be zero on the separation plane $(z = 0)$. We have

$$\begin{aligned} &\zeta_{1,1}(x - x^r) + \zeta_{1,2}(y - y^r) + \zeta_{1,3}h = 0, \\ &\zeta_{2,1}(x - x^r) + \zeta_{2,2}(y - y^r) + \zeta_{2,3}h = 0. \end{aligned} \tag{24}$$

Solving this equation and using x_0^q and y_0^q to indicate the confocal point on the plane $z = 0$, the following equation is obtained

$$x_0^q = -\frac{(\zeta_{2,2}\zeta_{1,3} - \zeta_{1,2}\zeta_{2,3})h}{\zeta_{2,2}\zeta_{1,1} - \zeta_{1,2}^2} + x^r, \qquad y_0^q = -\frac{(\zeta_{1,1}\zeta_{2,3} - \zeta_{1,2}\zeta_{1,3})h}{\zeta_{2,2}\zeta_{1,1} - \zeta_{1,2}^2} + y^r. \tag{25}$$

Because $R(x^r, y^r, z^r)$ is the reflection image of the source $P(x^p, y^p, z^p)$ on the horizontal plane $z = 0$, we must have

$$x^q = x_0^q, \qquad y^q = y_0^q. \tag{26}$$

Substituting (20) and (25) in (26) and noting (21) and (23), we can obtain the image-source point coordinates, which are

$$x^r = 2h\gamma_1 + x^p, \qquad y^r = 2h\gamma_2 + y^p, \qquad z^r = -h. \tag{27}$$

4.3. Boundary condition

Though now we use the image method to obtain an initial solution, this solution must satisfy the boundary conditions, *i.e.* (5) and (6). For convenience of validating the air–earth boundary condition, we must first prove the following relationship on the boundary $(z = 0)$.

$$|\eta_{z=0}| = |\eta'_{z=0}|, \tag{28}$$

where the $|\eta|$ and $|\eta'|$ are given as (13) and (18). For simplicity, we assume $x^p = y^p = 0$. When we expand this equation, we obtain the following.

$$\begin{aligned} &2x(\zeta_{1,1}x^r + \zeta_{1,2}y^r - 2\zeta_{1,3}h) + 2y(\zeta_{1,2}x^r + \zeta_{2,2}y^r - 2\zeta_{2,3}h) \\ &\quad + \zeta_{1,1}x^{r2} + \zeta_{2,2}y^{r2} + 2\zeta_{1,2}x^r y^r - 2\zeta_{1,3}x^r h - 2\zeta_{2,3}y^r h = 0, \end{aligned} \tag{29}$$

where the x and y are arbitrary variables on the air–earth surface. Consequently, the following equations must be satisfied

$$\zeta_{1,1}x^r + \zeta_{1,2}y^r - 2\zeta_{1,3}h = 0, \tag{30}$$

$$\zeta_{1,2}x^r + \zeta_{2,2}y^r - 2\zeta_{2,3}h = 0, \tag{31}$$

$$\zeta_{1,1}x^{r2} + \zeta_{2,2}y^{r2} + 2\zeta_{1,2}x^r y^r - 2\zeta_{1,3}x^r h - 2\zeta_{2,3}y^r h = 0. \tag{32}$$

Substituting (27) in (30), (31), (32) and noting $x^p = y^p = 0$, we can confirm their validity.

Taking the partial derivatives of (17) with respect to the spatial coordinates x, y and z, respectively, setting $z = 0$ and using condition (28), we may obtain the following equations

$$\frac{\partial v}{\partial x} = [\zeta_{1,1}(x - x^p) + \zeta_{2,1}(y - y^p) - \zeta_{1,1}h\gamma_1 - \zeta_{2,1}h\gamma_2]I'/[\pi(|\eta|)^3], \tag{33}$$

$$\frac{\partial v}{\partial y} = [\zeta_{1,2}(x - x^p) + \zeta_{2,2}(y - y^p) - \zeta_{1,2}h\gamma_1 - \zeta_{2,2}h\gamma_2]I'/[\pi(|\eta|)^3], \tag{34}$$

$$\frac{\partial v}{\partial z} = [\zeta_{1,3}(x - x^p) + \zeta_{2,3}(y - y^p) - \zeta_{1,3}h\gamma_1 - \zeta_{2,3}h\gamma_2]I'/[\pi(|\eta|)^3]. \tag{35}$$

Substituting (33), (34) and (35) in (5) we have

$$\begin{aligned} &(\sigma_{1,3}\zeta_{1,1} + \sigma_{2,3}\zeta_{1,2} + \sigma_{3,3}\zeta_{1,3})[(x - x^p) + h\gamma_1] \\ &\quad + (\sigma_{1,3}\zeta_{2,1} + \sigma_{2,3}\zeta_{2,2} + \sigma_{3,3}\zeta_{2,3})[(y - y^p) + h\gamma_2] \equiv 0, \end{aligned} \tag{36}$$

where $(x - x^p) + h\gamma_1$ and $(y - y^p) + h\gamma_2$ are not always zero on the xy-plane, so we must have

$$\sigma_{1,3}\zeta_{1,1} + \sigma_{2,3}\zeta_{1,2} + \sigma_{3,3}\zeta_{1,3} \equiv 0, \tag{37}$$

$$\sigma_{1,3}\zeta_{2,1} + \sigma_{2,3}\zeta_{2,2} + \sigma_{3,3}\zeta_{2,3} \equiv 0. \tag{38}$$

Substituting $\zeta_{i,j}$ $(\zeta_{i,j} = (\sigma_{i,j})^{-1})$ in (37) and (38), we can easily validate them.

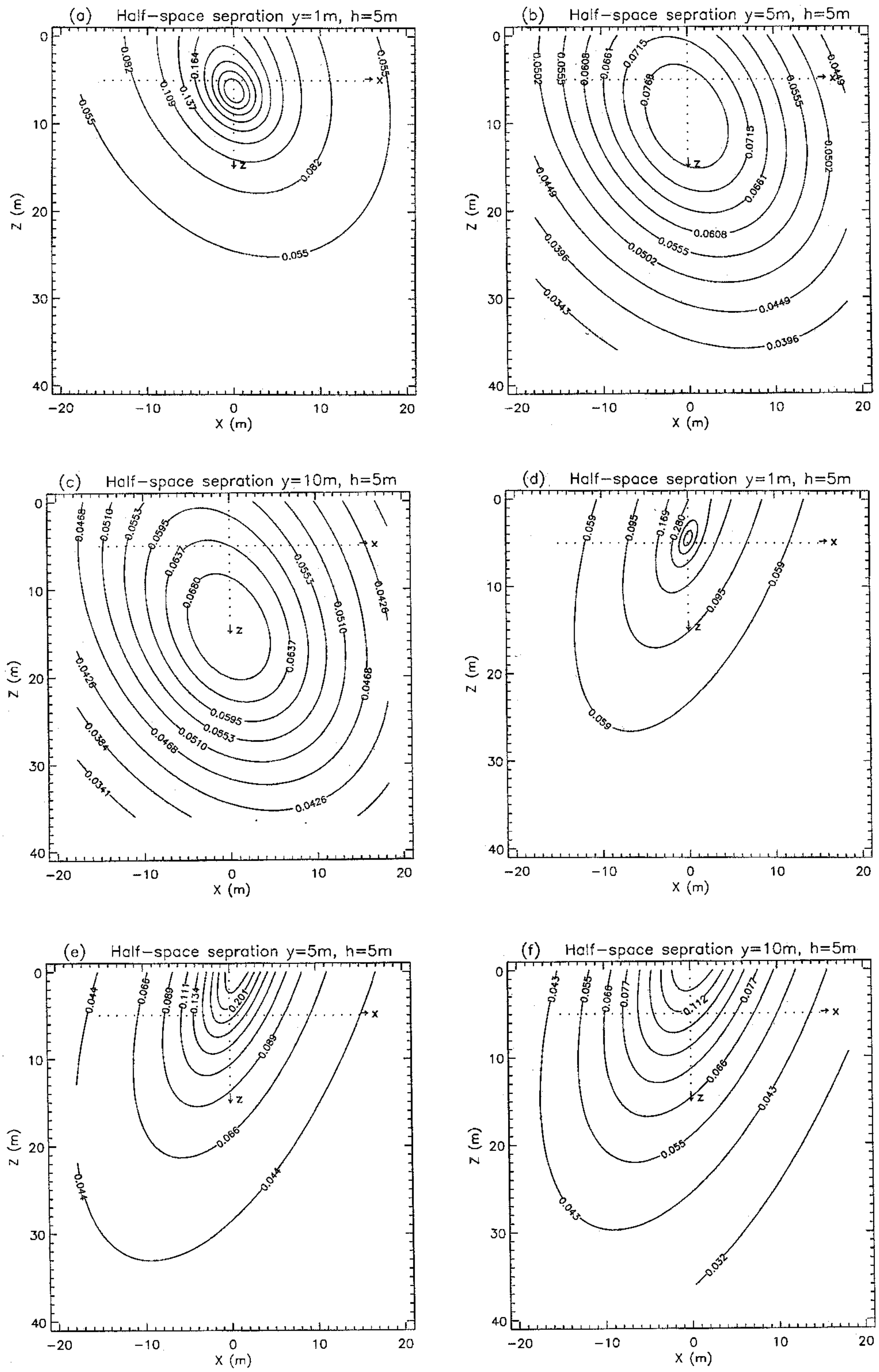

Figure 6. Equipotential contours arising from a point source in an arbitrarily anisotropic medium. Figure 6(a, b and c) are the equipotentials with tensor σ^1 on separations plane $y = 1, 5, 10\,\mathrm{m}$ respectively and (d, e and f) are the corresponding equipotentials of tensor σ^2.

5. Numerical results

Two representative conductivity tensors were selected to demonstrate the theoretical solution presented in this paper,

$$\text{(a)}\quad \sigma^1_{i,j} = \begin{pmatrix} 0{\cdot}3 & 0{\cdot}005 & 0{\cdot}1 \\ 0{\cdot}005 & 0{\cdot}06 & 0{\cdot}08 \\ 0{\cdot}1 & 0{\cdot}08 & 0{\cdot}5 \end{pmatrix}, \qquad \text{(b)}\quad \sigma^2_{i,j} = \begin{pmatrix} 0{\cdot}080 & 0{\cdot}005 & -0{\cdot}08 \\ 0{\cdot}005 & 0{\cdot}2 & -0{\cdot}1 \\ -0{\cdot}08 & -0{\cdot}1 & 0{\cdot}3 \end{pmatrix}.$$

Let an electric source at point $P(0, 0, h)$ inject direct current into an anisotropic half-space. Using (27), we can calculate the values of the coordinates x^r, y^r and z^r of the image-source point. And then, using (17), we can obtain the potential values at any point of the medium. In Figures 4 and 5 the equipotential contours due to a point source located at various depths are shown. Figures 4 and 5(a, b and c) are the whole space potential for different values of h (5, 20 and 40 m), and Figures 4 and 5(d, e and f) are the corresponding half-spaces solutions on the surface of the earth medium. The point source is located vertically below the crossing of the two dotted lines on each horizontal plane. Obviously, the *shapes* of the half-space contours on the air–earth surface are the same as for the whole space when the value of h is the same. But the *values* of potential in the half-space surface are twice those for the whole space case. The centre of the concentric equipotential contours moves horizontally away from the source. The main controlling factors in this are the tensor elements $\sigma_{3,1}$ and $\sigma_{3,2}$. For example, when the tensor element $(\sigma^1_{3,1})$ is positive, the concentric equipotential contours on the air–earth surface move along the negative x-direction, and when the tensor element $(\sigma^2_{3,1})$ is negative the concentric equipotential contours on the air–earth surface move in the positive x-direction. It is also found that the concentric equipotential contours move along the x-direction more than in the y-direction, because the absolute value of the element $\sigma_{3,1}$ is greater than the absolute value of the element $\sigma_{3,2}$.

In Figure 6 we show the equipotential contours due to a point source located at 5 m. The (a) (b) and (c) are the potentials for tensor $\sigma^1_{i,j}$ for different separations $y = 1, 5, 10$ m and the (d), (e) and (f) are the potentials for tensor $\sigma^2_{i,j}$ when the separations are the same as the first case. We find that different tensors produce different equipotential contours. The orientation, contour gradients and the concentric equipotential contours are different. The confocal point for tensor $\sigma^1_{i,j}$ comes down along the positive z axis, but the confocal point for tensor $\sigma^2_{i,j}$ moves up away from the air–earth surface.

6. Summary

An analytical solution for the point-electric-current-source potential in an arbitrarily anisotropic half-space was derived by extending the conventional method of images to the arbitrarily anisotropic case, and we used a linear coordinate transformation and traditional extreme value theory. When the image-source method is used in an arbitrarily anisotropic half-space, the key problem is the determination of the image-source-point location. In this paper, we overcame this problem by adopting the following three steps. Firstly, a linear coordinate transformation (rotation and translation) is applied and the elliptical partial differential equation (4) becomes Poisson's Equation (10), so that a very similar solution to the case of a point source in a homogeneous isotropic medium was obtained as (12). Secondly, based on the assumption that an image-source point exists somewhere and in analogy with the isotropic and homogeneous case we take the half-space analytical solution as (17). Thirdly, traditional extreme-value theory is applied to determine the image-source point position which is given in (27).

To demonstrate conveniently how the boundary conditions (5) are satisfied, we proved the equality of $|\eta|$ and $|\eta'|$ on the air–earth surface, using the expression for the image-point-source position. Using this relationship, we readily met the boundary condition.

The solution provides a complete description of the potential in the half-space earth with a truly arbitrarily anisotropic three-dimensional medium. Two different conductivity tensors have been used to determine the potential and the equipotential contours depicted in Figures 4 and 5. These clearly show the influence of anisotropy in the medium from the shapes of the equipotential contours.

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